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GUGGENHEIM AERONAUTICAL LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY

HYPERSONIC RESEARCH PROJECT

Memorandum No. 59
September 20, 1960

SMALL PERTURBATIONS IN THE UNSTEADY FLOW OF A RAREFIED GAS BASED ON GRAD'S THIRTEEN MOMENT APPROXIMATION

by
Daniel Kwoh-i Ai

ARMY ORDNANCE CONTRACT NO. DA-04-495-Ord-1960

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ABSTRACT

In this paper, the unsteady one-dimensional flow of a compressible, viscous and heat conducting fluid is treated, based on linearized Grad's thirteen moment equations. The fluid, initially at rest, is set into motion by some small external disturbances. Our interest is to examine the nature of all the responses. The fluid field extends to infinity in both directions; thus no length is involved, and also there is no solid wall boundary existing in the problem. The nature of the external disturbances is restricted to having a unit impulse in the momentum equation and a unit heat addition in the energy equation. The disturbances are located on an infinite plane normal to the flow direction; and the responses induced correspond to fundamental solutions of the problem. The method of Laplace transforms is applied, and the inverse transforms of all quantities are obtained in integral form. Because of the complicated expressions of the integrands involved, we consider only certain limiting cases which correspond to small and large times from the start of the motion, compared to the average time between molecular collisions. In order to study these limiting cases, it is essential to understand the behavior of the integrand in the complex plane; hence all singularities and branch points are obtained.

When t is small, the integrand is expanded in powers of t to obtain a wave front approximation. All discontinuities are propagated along the characteristics of the linearized system, and a damping term also appears.

At large values of time, the integrand gets its main contribution around the branch points, and these solutions are identical to those ob-

tained from the Navier-Stokes equation.

The fundamental solution of the one-dimensional unsteady flow, idealized as it seems to be, offers itself as a tool to understand other related problems. The piston problem, as well as the normal quantities in Rayleigh's problem (e. g., normal velocity, normal stress, and thermodynamical quantities), are governed by the same set of equations. Hence, certain parts of the fundamental solutions can be applied directly to these problems. The limiting forms of the normal quantities in Rayleigh's problem are expected to be worked out in another paper in the near future.

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LIST OF SYMBOLS

c	isentropic speed of sound, $c^2 = p_o/\rho_o$
c_o	isothermal speed of sound, $c_o^2 = p_o/\rho_o$
c_p	specific heat at constant pressure
F	external force
$G^{(i)}$	Green's functions
H	heat addition
K	coefficient of heat conductivity
P	hydrodynamic pressure
p	non-dimensional perturbation pressure
p_o	hydrodynamic pressure of the fluid at rest
q	heat flux
s	non-dimensional perturbation density
$\mathscr{s}$	Laplace transform variable
T	temperature
T_o	temperature of the fluid at rest
x, t	distorted space and time coordinates as defined in the text
x', t'	physical space and time coordinates
γ	ratio of specific heats at constant pressure and constant volume
$\delta(x)\delta(t)$	delta functions, such that $\iint \delta(x)\delta(t)\, dx\, dt = 1$
θ	non-dimensional perturbation temperature
μ	coefficient of viscosity
μ_o	coefficient of viscosity of the fluid at rest
ρ	density
ρ_o	density of the fluid at rest
p_{ij}, τ	stress increment over the hydrodynamic pressure

I. INTRODUCTION

Grad's thirteen moment equations, derived from kinetic theory considerations, represent a formidable set of non-linear equations far more complicated than any set of the hydrodynamic equations one usually encounters. It would seem reasonable to tackle the simplest possible problems first. If one examines the solutions of Grad's equations in existence, one is not surprised to find that most cases considered so far are linearized(1, 6, 10, 11), and involve very simple geometry(5). It has been known for a long time that linearized hydrodynamic equations offer solutions of such a nature that one obtains not only the overall picture, but also some typical features of the exact non-linear problems are still retained(7). The linearization may also be justified by saying that it makes mathematical treatment possible, and thus allows one to carry out a unified discussion of various effects(3, 8, 9). Furthermore, within the frame of linear theory, superposition can always be applied to construct new solutions. For these reasons, a similar treatment is attempted for Grad's equations.

So far, the solutions obtained for Grad's equations are all for the steady state case, except Rayleigh's problem(11) treated by Yang and Lees. In that particular problem, equations of "acoustic" nature and solutions at least in limiting cases are obtained for a heat insulated plate. The characteristics show an initial linear growth in time, and the solutions show interesting features which are quite different in nature to that of Navier-Stokes(8). It was also suggested that more non-stationary problems should be taken up for investigation. The present work concerns the one-dimensional unsteady problem, which may be considered as an extension of Rayleigh's problem (normal quantities). On the other hand, it bears a certain resemblance to the piston problem. In both cases the longitudinal waves[7] play an important part.

The fundamental solutions of the problem are the main interest in the present work. Since there is no solid boundary involved, the solutions are relatively simple to obtain. Furthermore, the introduction of impulse functions makes all solutions appear as contour integrals; consequently, studies of limiting cases can be carried out without too much difficulty. Although the problem seems to be quite idealized, solutions obtained yield appreciable amounts of information that are useful in considering other cases, such as the piston or heat conduction of an infinite plate.

II. LINEARIZED GRAD'S EQUATIONS

The general Grad's thirteen moment equations(4) with external force and heat addition are given below in Cartesian tensor form:

Continuity

$$\frac{\partial \rho}{\partial t'} + \frac{\partial}{\partial x'_\alpha}(\rho u_\alpha) = 0 \tag{1}$$

Momentum

$$\frac{\partial u_i}{\partial t'} + u_\alpha \frac{\partial u_i}{\partial x'_\alpha} + \frac{1}{\rho}\frac{\partial p_{i\alpha}}{\partial x'_\alpha} + \frac{1}{\rho}\frac{\partial P}{\partial x'_i} = F_i' \tag{2}$$

Energy

$$\frac{\partial P}{\partial t'} + \frac{\partial}{\partial x'_\alpha}(u_\alpha P) + \frac{2}{3}(p_{i\alpha} + \delta_{i\alpha}P)\frac{\partial u_i}{\partial x'_\alpha} + \frac{2}{3}\frac{\partial q_\alpha}{\partial x'_\alpha} = H' \tag{3}$$

Stresses

$$\frac{\partial p_{ij}}{\partial t'} + \frac{\partial}{\partial x'_\alpha}(u_\alpha p_{ij}) + \frac{2}{5}\left(\frac{\partial q_i}{\partial x'_j} + \frac{\partial q_j}{\partial x'_i} - \frac{2}{3}\delta_{ij}\frac{\partial q_\alpha}{\partial x'_\alpha}\right) + p_{i\alpha}\frac{\partial u_j}{\partial x'_\alpha} \tag{4}$$

$$+ p_{j\alpha}\frac{\partial u_i}{\partial x'_\alpha} - \frac{2}{3}\delta_{ij}p_{\alpha\beta}\frac{\partial u_\alpha}{\partial x'_\beta} + P\left(\frac{\partial u_i}{\partial x'_j} + \frac{\partial u_j}{\partial x'_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_\alpha}{\partial x'_\alpha}\right) = -\frac{P}{\mu}p_{ij}$$

Heat Flux

$$\frac{\partial q_i}{\partial t'} + \frac{\partial}{\partial x'_\alpha}(u_\alpha q_i) + \frac{7}{5} q_\alpha \frac{\partial u_i}{\partial x'_\alpha} + \frac{2}{5} q_\alpha \frac{\partial u_\alpha}{\partial x'_i} + \frac{2}{5} q_i \frac{\partial u_\alpha}{\partial x'_\alpha} + RT \frac{\partial p_{i\alpha}}{\partial x'_\alpha}$$

$$+ \frac{7}{2} p_{i\alpha} \frac{\partial RT}{\partial x'_\alpha} + \frac{5}{2} P \frac{\partial RT}{\partial x'_i} - \frac{p_{i\alpha}}{\rho}\left(\frac{\partial p_{\alpha\beta}}{\partial x'_\beta} + \delta_{\alpha\beta}\frac{\partial P}{\partial x'_\beta}\right) = -\frac{2}{3}\frac{P}{\mu} q_i \tag{5}$$

Altogether, fifteen unknowns are involved in equations (1) to (5); hence, we need in addition the equation of state, which is also obtained from certain moment relation[4], to complete the set.

$$P = \rho R T \tag{6}$$

Furthermore, from the definition of the moments and also from equation (4), there exists the relation

$$p_{ii} = 0$$

Therefore, in general, only five stresses are to be solved, and the total number of moment equations reduces to thirteen.

In the following, the analysis will be based on the theory of small perturbations. By small perturbations, we mean that

$$P = p_0(1+p) \quad , \quad \rho = \rho_0(1+s) \quad , \quad T = T_0(1+\theta) \tag{7}$$

where p, θ, $s \ll 1$ everywhere, and $|\vec{u}| \ll c$, c being the isentropic speed of sound.

For stresses and heat flux, we have

$$p_{ij}/p_o << 1 \qquad \text{and} \qquad q_i \sim O(p_o u_i)$$

We also assume

$$\mu = \mu_o(1+\mu') \tag{8}$$

where $\mu_o = \mu_o(T_o)$ and $\mu' << 1$

We can utilize the above relations to linearize equations (1) to (6) by dropping all products and squares of perturbations. A set of linearized equations is obtained in the following form.

Continuity

$$\frac{\partial s}{\partial t'} + \frac{\partial u_\alpha}{\partial x_\alpha'} = 0 \tag{9}$$

Momentum

$$\frac{\partial u_i}{\partial t'} + \frac{1}{\rho_o}\frac{\partial p_{i\alpha}}{\partial x_\alpha'} + \frac{p_o}{\rho_o}\frac{\partial p}{\partial x_i'} = F_i' \tag{10}$$

Energy

$$p_o\frac{\partial p}{\partial t'} + p_o\frac{\partial u_\alpha}{\partial x_\alpha'} + \frac{2}{3}p_o\delta_{i\alpha}\frac{\partial u_i}{\partial x_\alpha'} + \frac{2}{3}\frac{\partial q_\alpha}{\partial x_\alpha'} = H' \tag{11}$$

Stresses

$$\frac{\partial p_{ij}}{\partial t'} + \frac{2}{5}\left(\frac{\partial q_i}{\partial x_j'} + \frac{\partial q_j}{\partial x_i'} - \frac{2}{3}\delta_{ij}\frac{\partial q_\alpha}{\partial x_\alpha'}\right)$$

$$+ p_0\left(\frac{\partial u_i}{\partial x_j'} + \frac{\partial u_j}{\partial x_i'} - \frac{2}{3}\delta_{ij}\frac{\partial u_\alpha}{\partial x_\alpha'}\right) = -\frac{p_0}{\mu_0}p_{ij} \qquad (12)$$

Heat Flux

$$\frac{\partial q_i}{\partial t'} + RT_0\frac{\partial p_{i\alpha}}{\partial x_\alpha'} + \frac{5}{2}p_0 RT_0\frac{\partial \theta}{\partial x_i'} = -\frac{2}{3}\frac{p_0}{\mu_0}q_i \qquad (13)$$

State

$$p = s + \theta \qquad (14)$$

The kind of linearization used above is very common in hydrodynamics. One would get the steady state Oseen's type of equations by applying a Galilean transformation to the above equations[7].

III. ONE-DIMENSIONAL UNSTEADY FLOW

III A. Equations of Motion

For the one-dimensional flow problem, the number of moments required is greatly reduced. Here, we have a set of 5 first order partial differential equations instead of the thirteen needed for the general case, and one algebraic (equation of state) equation. The six unknowns to be determined are the following:

p	perturbation pressure
s	perturbation density
θ	perturbation temperature
u	velocity
τ	normal stress
q	heat flux

The quantities p, s, and θ are non-dimensional. If we introduce a new set of coordinates

$$x = \frac{p_0}{\mu_0} x' \quad , \quad t = \frac{p_0}{\mu_0} t'$$

the six equations describing the flow are:

Continuity

$$\frac{\partial s}{\partial t} + \frac{\partial u}{\partial x} = 0 \tag{15}$$

Momentum

$$\frac{\partial u}{\partial t} + \frac{p_0}{\rho_0}\frac{\partial p}{\partial x} + \frac{1}{\rho_0}\frac{\partial \tau}{\partial x} = F(x,t) \tag{16}$$

Energy

$$\frac{\partial p}{\partial t} + \frac{5}{3}\frac{\partial u}{\partial x} + \frac{2}{3p_0}\frac{\partial q}{\partial x} = H(x,t) \tag{17}$$

Stress

$$\left(\frac{\partial}{\partial t} + 1\right)\tau + \frac{8}{15}\frac{\partial q}{\partial x} + \frac{4}{3}p_0\frac{\partial u}{\partial x} = 0 \tag{18}$$

Heat Flux

$$\left(\frac{\partial}{\partial t} + \frac{2}{3}\right)q + \frac{p_0}{\rho_0}\frac{\partial \tau}{\partial x} + \frac{5}{2}\frac{p_0^2}{\rho_0}\frac{\partial \theta}{\partial x} = 0 \tag{19}$$

State

$$p = s + \theta \tag{20}$$

where

$$F' = \frac{p_0}{\mu_0}F \qquad\qquad H' = \frac{p_0^2}{\mu_0}H$$

III B. Laplace Transforms with Zero Initial Conditions

Since our purpose is to determine the responses generated by small disturbances in a fluid field originally at rest, we may set $t = 0$ as the time at which the disturbances are introduced; therefore, only solutions for $t > 0$ are of interest to us. Hence, the method of Laplace transforms should bring out all solutions, at least in integral form.

The Laplace transform with respect to t of any quantity $Q = Q(x,t)$ is

$$\mathcal{L}\{Q\} = \bar{Q} = \int_0^\infty e^{-st} Q(x,t)\, dt \tag{21}$$

With zero initial condition,

$$\mathcal{L}\left\{\frac{\partial Q}{\partial t}\right\} = s\bar{Q} \tag{22}$$

and the inverse transform is defined as

$$Q = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} e^{st}\,\bar{Q}\, ds \tag{23}$$

where σ is to the right of all singularities. The transformed equations become ordinary differential equations of the independent variable, x, as follows:

Continuity

$$\frac{d\bar{u}}{dx} = -s\bar{S} \tag{24}$$

Momentum

$$s\bar{u} + \frac{p_0}{\rho_0}\frac{d\bar{p}}{dx} + \frac{1}{\rho_0}\frac{d\bar{\tau}}{dx} = \bar{F}(x;s) \tag{25}$$

Energy

$$\frac{d\bar{q}}{dX} = \frac{3}{2} p_0 s \left(\frac{5}{3} \bar{s} - \bar{p} \right) + \frac{3}{2} p_0 \bar{H}(x; s) \tag{26}$$

Stress

$$\bar{\tau} = \frac{4}{5} p_0 \frac{s}{s+1} \bar{p} \tag{27}$$

Heat

$$\left(s + \frac{2}{3}\right) \bar{q} + \frac{p_0}{\rho_0} \frac{d\bar{\tau}}{dX} + \frac{5}{2} \frac{p_0^2}{\rho_0} \frac{d\bar{\theta}}{dX} = 0 \tag{28}$$

State

$$\bar{p} = \bar{s} + \bar{\theta} \tag{29}$$

III C. Solutions of Transformed Equations

In order to solve the six unknowns from Eqs. (24) to (29), we start off by eliminating $\bar{\theta}$ from (29), $\bar{u}$ from (24), $\bar{\tau}$ from (27), and $\bar{q}$ from (28) successively to arrive at two simultaneous equations for $\bar{s}$ and $\bar{p}$.

$$-s^2(s+1)\bar{s} + \frac{p_0}{\rho_0}\left(\frac{9}{5} s + 1\right) \frac{d^2 \bar{p}}{dX^2} = (s+1) \frac{d\bar{F}}{dX} \tag{30}$$

$$\left[\frac{p_0}{\rho_0}(s+1)\frac{d^2}{dX^2} - s(s+1)\left(s + \frac{2}{3}\right)\right]\bar{s} - \left[\frac{p_0}{\rho_0}\left(\frac{33}{25} s + 1\right)\frac{d^2}{dX^2} - \frac{3}{5} s (s+1)\left(s + \frac{2}{3}\right)\right]\bar{p}$$
$$= \frac{3}{5}(s+1)\left(s + \frac{2}{3}\right)\bar{H} \tag{31}$$

Cross-differentiation of the above equations yields the governing equations of $\bar{s}$ and $\bar{p}$ respectively.

$$\frac{p_0^2}{\rho_0^2}\left(\frac{9}{5}s+1\right)\frac{d^4\bar{s}}{dx^4} - \frac{p_0}{\rho_0}s\left(\frac{78}{25}s^2+\frac{16}{5}s+\frac{2}{3}\right)\frac{d^2\bar{s}}{dx^2}$$

$$+\frac{3}{5}s^3(s+1)(s+\tfrac{2}{3})\bar{s} = \frac{p_0}{\rho_0}\left(\frac{33}{25}s+1\right)\frac{d^3\bar{F}}{dx^3}$$

$$-\frac{3}{5}s(s+1)(s+\tfrac{2}{3})\frac{d\bar{F}}{dx} + \frac{3}{5}\frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)(s+\tfrac{2}{3})\frac{d^2\bar{H}}{dx^2} \qquad (32)$$

$$\frac{p_0^2}{\rho_0^2}\left(\frac{9}{5}s+1\right)\frac{d^4\bar{p}}{dx^4} - \frac{p_0}{\rho_0}s\left(\frac{78}{25}s^2+\frac{16}{5}s+\frac{2}{3}\right)\frac{d^2\bar{p}}{dx^2}$$

$$+\frac{3}{5}s^3(s+1)(s+\tfrac{2}{3})\bar{p} = \frac{p_0}{\rho_0}(s+1)\frac{d^3\bar{F}}{dx^3}$$

$$-s(s+1)(s+\tfrac{2}{3})\frac{d\bar{F}}{dx} + \frac{3}{5}s^2(s+1)(s+\tfrac{2}{3})\bar{H} \qquad (33)$$

We notice that both equations have the same homogeneous part, but the inhomogeneous parts differ. In fact, the same linear differential operator governs all unknown quantities to be determined. This behavior is expected, since the linear operator is related to the characteristics of the linearized system. To save writing, we denote

$$a = \frac{p_0^2}{\rho_0^2}\left(\frac{9}{5}s+1\right)$$

$$b = -\frac{p_0}{\rho_0}s\left(\frac{39}{25}s^2+\frac{8}{5}s+\frac{1}{3}\right)$$

$$c = \frac{3}{5} s^3 (s+1)(s+\tfrac{2}{3})$$

$$f_{\bar{s}} = f_{\bar{s}_F} + f_{\bar{s}_H}$$

$$f_{\bar{p}} = f_{\bar{p}_F} + f_{\bar{p}_H}$$

where

$$f_{\bar{s}_F} = -\frac{(\frac{33}{25}s+1)}{\frac{p_o}{\rho_o}(\frac{9}{5}s+1)} \frac{d^3\bar{F}}{dx^3} + \frac{\frac{3}{5}s(s+1)(s+\frac{2}{3})\rho_o/p_o}{\frac{p_o}{\rho_o}(\frac{9}{5}s+1)} \frac{d\bar{F}}{dx}$$

$$f_{\bar{s}_H} = -\frac{3}{5}(s+\tfrac{2}{3}) \frac{d^2\bar{H}}{dx^2}$$

$$f_{\bar{p}_F} = -\frac{(s+1)}{\frac{p_o}{\rho_o}(\frac{9}{5}s+1)} \frac{d^3\bar{F}}{dx^3} + \frac{s(s+1)(s+\frac{2}{3})}{\frac{p_o}{\rho_o}(\frac{9}{5}s+1)} \frac{\rho_o}{p_o} \frac{d\bar{F}}{dx}$$

$$f_{\bar{p}_H} = -\frac{\frac{3}{5}s^2(s+1)(s+\frac{2}{3})}{\frac{p_o}{\rho_o}(\frac{9}{5}s+1)} \bar{H}$$

The quantities $f_{\bar{s}_F}$ and $f_{\bar{p}_F}$ are introduced by the external forcing term, while $f_{\bar{s}_H}$ and $f_{\bar{p}_H}$ correspond to the heat addition term. Since the equations are linear, the solutions associated with $\bar{F}$ and $\bar{H}$ can be treated separately.

Equations (32) and (33) are now written as

$$a\left(\frac{d^2}{dX^2}-\lambda_1\right)\left(\frac{d^2}{dX^2}-\lambda_2\right)\bar{s},\bar{p} = -(f_{\bar{s}}, f_{\bar{p}})\,a \tag{34}$$

Here we already have factored the fourth order operator

$$a\frac{d^4}{dX^4} + 2b\frac{d^2}{dX^2} + c$$

into the product of two second-order operators, $\left(\frac{d^2}{dX^2} - \lambda_1\right)$ and $\left(\frac{d^2}{dX^2} - \lambda_2\right)$. The λ' are given below.

$$\lambda_{1,2} = -\frac{b}{a} \pm \frac{1}{a}\sqrt{b^2 - ac} \tag{35}$$

or more precisely,

$$\lambda_{1,2} = \frac{s}{\frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)}\left[\frac{39}{25}s^2 + \frac{8}{5}s + \frac{1}{3} \pm \sqrt{\frac{846}{625}s^4 + \frac{324}{125}s^3 + \frac{47}{25}s^2 + \frac{2}{3}s + \frac{1}{9}}\right] \tag{35a}$$

The λ's are identical to the quantity $g(s)\{f_1(s) \pm f_2(s)\}$ obtained by Yang and Lees in reference (2), except that here distorted coordinates are used, so that the s differs by a factor of p_0/μ_0.

So far, the forcing functions F and H are left open. They can be any well-behaved functions. However, our present interest is to find the fundamental solutions; thus, we specify

$$F(x,t) = \delta(x)\delta(t) \tag{36}$$

and

$$H(x,t) = \delta(x)\delta(t) \tag{37}$$

The term F(x, t) represents a unit impulse in the x , t plane distributed evenly on an infinite plane normal to the x-axis at $x = 0$ and $t = 0$. This is equivalent to a uniform impulse of strength μ_o^3/p_o^3 in the x' , t' plane located at $x' = 0$ and $t' = 0$. Similarly, the term H(x, t) represents a unit heat input introduced at $t = 0$ and at the plane $x = 0$. In the physical plane, the addition of heat is of the magnitude of μ_o^3/p_o^4. The integrations of F(x, t) and H(x, t) taken with respect to x and t through any interval including the origin are unity. The reasons that one is interested in the fundamental solutions are the following:

(1) In principle, having found the fundamental solutions of the problem, solutions corresponding to any other given functions can be generated. Furthermore, fundamental solutions themselves yield an appreciable amount of information.

(2) All solutions will appear in the form of contour integrals in the complex s plane. Either these integrations can be performed exactly, or certain limiting forms can be obtained if the integrands become too involved, which turns out to be the case in this problem.

The transforms of F and H are:

$$F(x;s) = \int_0^\infty e^{-st}\delta(x)\delta(t)\,dt = \delta(x) \tag{38}$$

and

$$H(x;s) = \int_0^\infty e^{-st}\delta(x)\delta(t)\,dt = \delta(x) \tag{39}$$

and

$$\frac{d^n}{dx^n}\bar{F} = \frac{d^n}{dx^n}\bar{H} = \frac{d^n\delta(x)}{dx^n} \tag{40}$$

is the n$^{\text{th}}$ order derivative of $\bar{F}$ or $\bar{H}$. Having specified the forms of $\bar{F}$ and $\bar{H}$, we can write down immediately the solutions of the equations (32) and (33).

$$\bar{S} = \bar{S}_F + \bar{S}_H$$

$$\bar{S}_F = \int_{-\infty}^{+\infty} G^{(2)}(x;\xi) f_{\bar{S}_F}(\xi)\, d\xi \,, \quad \bar{S}_H = \int_{-\infty}^{+\infty} G^{(2)}(x;\xi) f_{\bar{S}_H}(\xi)\, d\xi \tag{41}$$

$$\bar{P} = \bar{P}_F + \bar{P}_H$$

$$\bar{P}_F = \int_{-\infty}^{+\infty} G^{(2)}(x;\xi) f_{\bar{P}_F}(\xi)\, d\xi \,, \quad \bar{P}_H = \int_{-\infty}^{+\infty} G^{(2)}(x;\xi) f_{\bar{P}_H}(\xi)\, d\xi \tag{42}$$

where

$$G^{(2)}(x;\xi) = G^{(2)}(x-\xi) = \frac{1}{\lambda_1 - \lambda_2}\left(G_1^{(1)} - G_2^{(1)}\right)$$

$$G_1^{(1)}(x;\xi) = G_1^{(1)}(x-\xi) = \frac{1}{2\sqrt{\lambda_1}}\, e^{-\sqrt{\lambda_1}|x-\xi|}$$

$$G_2^{(1)}(x;\xi) = G_2^{(1)}(x-\xi) = \frac{1}{2\sqrt{\lambda_1}}\, e^{-\sqrt{\lambda_1}|x-\xi|}$$

are the Green's functions of the operators $\left(\frac{d^2}{dx^2}-\lambda_1\right)\left(\frac{d^2}{dx^2}-\lambda_2\right)$, $\left(\frac{d^2}{dx^2}-\lambda_1\right)$ and $\left(\frac{d^2}{dx^2}-\lambda_2\right)$, respectively. Substituting the expressions of $f_{\bar{S}}$ and $f_{\bar{P}}$ into their corresponding equations, we obtain

$$\bar{S}_F = -\frac{\left(\frac{33}{25}\ell+1\right)}{\frac{p_0}{\rho_0}\left(\frac{9}{5}\ell+1\right)}\int_{-\infty}^{+\infty} G^{(2)}(x-\xi)\,\frac{d^3\delta(\xi)}{d\xi^3}\,d\xi$$
$$+\frac{\frac{3}{5}\ell(\ell+1)(\ell+\frac{2}{3})\rho_0/p_0}{\frac{p_0}{\rho_0}\left(\frac{9}{5}\ell+1\right)}\int_{-\infty}^{+\infty} G^{(2)}(x-\xi)\,\frac{d\delta(\xi)}{d\xi}\,d\xi \tag{43}$$

$$\bar{S}_H = -\frac{3}{5}\left(\ell+\frac{2}{3}\right)\int_{-\infty}^{+\infty} G^{(2)}(x-\xi)\,\frac{d^2\delta(\xi)}{d\xi^2}\,d\xi \tag{44}$$

$$\bar{P}_F = -\frac{(\ell+1)}{\frac{p_0}{\rho_0}(\frac{9}{5}\ell+1)}\int_{-\infty}^{+\infty} G^{(2)}(x-\xi)\,\frac{d^3\delta(\xi)}{d\xi^3}\,d\xi$$

$$+\frac{\ell(\ell+1)(\ell+\frac{2}{3})\frac{\rho_0}{p_0}}{\frac{p_0}{\rho_0}(\frac{9}{5}\ell+1)}\int_{-\infty}^{+\infty} G^{(2)}(x-\xi)\frac{d\delta(\xi)}{d\xi}\,d\xi \tag{45}$$

$$\bar{P}_H = -\frac{\frac{3}{5}\ell^2(\ell+1)(\ell+\frac{2}{3})}{\frac{p_0}{\rho_0}(\frac{9}{5}\ell+1)}\int_{-\infty}^{+\infty} G^{(2)}(x-\xi)\,\delta(\xi)\,d\xi \tag{46}$$

One would expect that in order to obtain all the transformed quantities, the integral

$$I_n = \int_{-\infty}^{+\infty} G^{(2)}(x-\xi)\,\frac{d^n\delta(\xi)}{d\xi^n}\,d\xi \tag{47}$$

for $n = 0$, 1, 2, and 3 must be evaluated. At $x - \xi$, the n^{th} derivative of the Green's function $G^{(2)}(x-\xi)$ with respect to ξ is continuous; therefore we can integrate Eq. (47) by parts. The results are collected in Appendix I.

We now have

$$\bar{S}_F = \frac{(\mathrm{sgn}\,x)\left[(\frac{33}{25}\ell+1)\lambda_1-\frac{3}{5}\frac{\rho_0}{p_0}\ell(\ell+1)(\ell+\frac{2}{3})\right]e^{-\sqrt{\lambda_1}|x|}}{2\frac{p_0}{\rho_0}(\frac{9}{5}\ell+1)(\lambda_1-\lambda_2)}+\frac{(\mathrm{sgn}\,x)\left[(\frac{33}{25}\ell+1)\lambda_2-\frac{3}{5}\frac{\rho_0}{p_0}\ell(\ell+1)(\ell+\frac{2}{3})\right]e^{-\sqrt{\lambda_2}|x|}}{2\frac{p_0}{\rho_0}(\frac{9}{5}\ell+1)(\lambda_2-\lambda_1)} \tag{48}$$

$$\bar{S}_H = -\frac{3/5(\ell+\frac{2}{3})\sqrt{\lambda_1}}{2(\lambda_1-\lambda_2)}e^{-\sqrt{\lambda_1}|x|} - \frac{3/5(\ell+\frac{2}{3})\sqrt{\lambda_2}}{2(\lambda_2-\lambda_1)}e^{-\sqrt{\lambda_2}|x|} \tag{49}$$

$$\bar{P}_F = \frac{(\mathrm{sgn}\,x)\left[(\ell+1)\lambda_1 - \frac{\rho_0}{\rho_0}\ell(\ell+1)(\ell+\frac{2}{3})\right]e^{-\sqrt{\lambda_1}|x|}}{2\frac{p_0}{\rho_0}(\frac{9}{5}\ell+1)(\lambda_1-\lambda_2)} + \frac{(\mathrm{sgn}\,x)\left[(\ell+1)\lambda_2 - \frac{\rho_0}{\rho_0}\ell(\ell+1)(\ell+\frac{3}{3})\right]e^{-\sqrt{\lambda_2}|x|}}{2\frac{p_0}{\rho_0}(\frac{9}{5}\ell+1)(\lambda_2-\lambda_1)} \tag{50}$$

$$\bar{P}_H = -\frac{3/5\,\ell^2(\ell+1)(\ell+\frac{2}{3})\frac{1}{\sqrt{\lambda_1}}e^{-\sqrt{\lambda_1}|x|}}{2\frac{p_0}{\rho_0}(\frac{9}{5}\ell+1)(\lambda_1-\lambda_2)} - \frac{3/5\,\ell^2(\ell+1)(\ell+\frac{2}{3})\frac{1}{\sqrt{\lambda_2}}e^{-\sqrt{\lambda_2}|x|}}{2\frac{p_0}{\rho_0}(\frac{9}{5}\ell+1)(\lambda_2-\lambda_1)} \tag{51}$$

From Eq. (41), we obtain

$$\bar{\theta}_F = -\frac{(\mathrm{sgn}\,x)\left[\frac{8}{25}\ell\lambda_1 + \frac{2}{5}\frac{\rho_0}{\rho_0}\ell(\ell+1)(\ell+\frac{2}{3})\right]e^{-\sqrt{\lambda_1}|x|}}{2\frac{p_0}{\rho_0}(\frac{9}{5}\ell+1)(\lambda_1-\lambda_2)} - \frac{(\mathrm{sgn}\,x)\left[\frac{8}{25}\ell\lambda_2 + \frac{2}{5}\frac{\rho_0}{\rho_0}\ell(\ell+1)(\ell+\frac{2}{3})\right]e^{-\sqrt{\lambda_2}|x|}}{2\frac{p_0}{\rho_0}(\frac{9}{5}\ell+1)(\lambda_2-\lambda_1)} \tag{52}$$

$$\bar{\theta}_H = -\frac{\frac{3}{5}(s+\frac{2}{3})\left[s^2(s+1)-\frac{p_0}{\rho_0}(\frac{9}{5}s+1)\lambda_1\right]e^{-\sqrt{\lambda_1}|x|}}{2\frac{p_0}{\rho_0}(\frac{9}{5}s+1)(\lambda_1-\lambda_2)\sqrt{\lambda_1}}$$

$$-\frac{\frac{3}{5}(s+\frac{2}{3})\left[s^2(s+1)-\frac{p_0}{\rho_0}(\frac{9}{5}s+1)\lambda_2\right]e^{-\sqrt{\lambda_2}|x|}}{2\frac{p_0}{\rho_0}(\frac{9}{5}s+1)(\lambda_2-\lambda_1)\sqrt{\lambda_2}} \tag{53}$$

With the aid of Eqs. (39), (40), and (37), the rest of the transforms are determined as

$$\bar{\tau}_F = \frac{4}{5}p_0\frac{(\operatorname{sgn}x)\left[s\lambda_1-\frac{\rho_0}{p_0}s^2(s+\frac{2}{3})\right]e^{-\sqrt{\lambda_1}|x|}}{2\frac{p_0}{\rho_0}(\frac{9}{5}s+1)(\lambda_1-\lambda_2)} + \frac{4}{5}p_0\frac{(\operatorname{sgn}x)\left[s\lambda_2-\frac{\rho_0}{p_0}s^2(s+\frac{2}{3})\right]e^{-\sqrt{\lambda_2}|x|}}{2\frac{p_0}{\rho_0}(\frac{9}{5}s+1)(\lambda_2-\lambda_1)} \tag{54}$$

$$\bar{\tau}_H = -\frac{12}{25}\frac{p_0\, s^3(s+\frac{2}{3})}{2\frac{p_0}{\rho_0}(\frac{9}{5}s+1)(\lambda_1-\lambda_2)}\frac{1}{\sqrt{\lambda_1}}e^{-\sqrt{\lambda_1}|x|} - \frac{12}{25}\frac{p_0\, s^3(s+\frac{2}{3})}{2\frac{p_0}{\rho_0}(\frac{9}{5}s+1)(\lambda_2-\lambda_1)}\frac{e^{-\sqrt{\lambda_2}|x|}}{\sqrt{\lambda_2}} \tag{55}$$

$$\bar{q}_F = -\frac{\rho_0\, s\sqrt{\lambda_1}\, e^{-\sqrt{\lambda_1}|x|}}{2(\lambda_1-\lambda_2)} - \frac{\rho_0\, s\sqrt{\lambda_2}\, e^{-\sqrt{\lambda_2}|x|}}{2(\lambda_2-\lambda_1)} \tag{56}$$

$$\bar{q}_H = -\frac{(\operatorname{sgn} x)\, p_0 \left\{ \frac{12}{25} s^3 + \frac{3}{2}\left[s^2(s+1) - \frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)\lambda_1 \right] \right\} e^{-\sqrt{\lambda_1}|x|}}{2\left(\frac{9}{5}s+1\right)(\lambda_1 - \lambda_2)}$$

$$- \frac{(\operatorname{sgn} x)\, p_0 \left\{ \frac{12}{25} s^3 + \frac{3}{2}\left[s^2(s+1) - \frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)\lambda_2 \right] \right\} e^{-\sqrt{\lambda_2}|x|}}{2\left(\frac{9}{5}s+1\right)(\lambda_2 - \lambda_1)} \tag{57}$$

$$\bar{u}_F = \frac{\left[\lambda_1 - \frac{\rho_0}{p_0} s\left(s+\frac{2}{3}\right)\right]\sqrt{\lambda_1}\, e^{-\sqrt{\lambda_1}|x|}}{2 s(\lambda_1 - \lambda_2)} + \frac{\left[\lambda_2 - \frac{\rho_0}{p_0} s\left(s+\frac{2}{3}\right)\right]\sqrt{\lambda_2}\, e^{-\sqrt{\lambda_2}|x|}}{2 s(\lambda_2 - \lambda_1)} \tag{58}$$

and

$$\bar{u}_H = -\frac{(\operatorname{sgn} x)\frac{p_0}{\rho_0}\frac{3}{5} s\left(s+\frac{2}{3}\right) e^{-\sqrt{\lambda_1}|x|}}{2\frac{p_0}{\rho_0}(\lambda_1 - \lambda_2)} - \frac{(\operatorname{sgn} x)\frac{3}{5} s\left(s+\frac{2}{3}\right) e^{-\sqrt{\lambda_2}|x|}}{2(\lambda_2 - \lambda_1)} \tag{59}$$

III D. Approximations

In the previous section IIIC, we have determined the transforms of all dependent variables. The exact evaluation of these transforms, however, involves a great deal of difficulty because of the complicated expressions we encounter. Nevertheless, certain approximations can be made without too much trouble. Furthermore, these approximations really represent limiting cases, which interest us. There are two approximations we consider in detail. One is the small time approximation; the other is the large time approximation.

By small times, we mean that the time elapsed from $t = 0$ is small compared with the average collision time t_f. By large values of time, we mean that the time elapsed from $t = 0$ is much larger than t_f. The physical significance of these approximations will be taken up again in Section IV.

1. Solutions Suitable for Small Values of Time. For small time, we are looking essentially for an expansion in powers of $1/s$.[2] By neglecting terms of the order of $1/s$, we have

$$\sqrt{\lambda_{1,2}} \cong \frac{s}{\sqrt{\frac{(13 \mp \sqrt{94})p_0}{5\rho_0}}} - \frac{5/9}{\sqrt{\frac{(13 \mp \sqrt{94})p_0}{5\rho_0}}}\left[\frac{1}{2} - \frac{1}{\sqrt{94}}\right] \tag{60}$$

The first term contributes in the integral

$$\exp\left[-\frac{s|x|}{\sqrt{\frac{(13 \mp \sqrt{94})p_0}{5\rho_0}}}\right]$$

which represents a translation through a distance of

$$|x| = \sqrt{\frac{(13 \mp \sqrt{94})\, p_0}{5\, \rho_0}}\; t$$

In order to understand this, let us now examine the characteristics of the equations.

The characteristics $\phi(x, t) =$ constant of the linearized system are found by the vanishing of the determinant:

$$\begin{vmatrix} \phi_t & 0 & \phi_x & 0 & 0 \\ 0 & \frac{p_0}{\rho_0}\phi_x & \phi_t & \frac{1}{\rho_0}\phi_x & 0 \\ 0 & \phi_t & \frac{5}{3}\phi_x & 0 & \frac{2}{3p_0}\phi_x \\ 0 & 0 & \frac{4}{3}p_0\phi_x & \phi_t & \frac{8}{15}\phi_x \\ -\frac{5}{2}\frac{p_0^2}{\rho_0}\phi_x & \frac{5}{2}\frac{p_0^2}{\rho_0}\phi_x & 0 & \frac{p_0}{\rho_0}\phi_x & \phi_t \end{vmatrix} = 0 \qquad (61)$$

Along $\phi(x, t) =$ constant

$$\frac{dx}{dt} = -\phi_t/\phi_x$$

where dx/dt is the slope of the various characteristic curves. The determinant Eq. (61) thus reduces to an algebraic equation for dx/dt.

$$\frac{dx}{dt}\left[\left(\frac{dx}{dt}\right)^4 - \frac{78}{15}\frac{p_0}{\rho_0}\left(\frac{dx}{dt}\right)^2 + 3\,\frac{p_0^2}{\rho_0^2}\right] = 0 \qquad (62)$$

With solutions

$$\frac{dx}{dt} = 0 \tag{63}$$

$$\frac{dx}{dt} = \pm\sqrt{\frac{(13 \pm \sqrt{94})}{5}\frac{p_0}{\rho_0}} \tag{64}$$

the solution $(dx/dt) = 0$ means that the particle path is one characteristic. The solutions of Eq. (64) represent characteristic directions at "sound speed" (dx/dt), different from the isentropic sound-speed which is the characteristic slope from the Euler equations. All the characteristics obtained here are identical to those given by Yang and Lees[11] corresponding to normal quantities. This behavior is to be expected, since no transverse quantity appears in the one-dimensional problem. Because of the linearization, all "characteristic curves" are straight lines and are known in advance. The characteristics normalized against the isothermal speed of sound are plotted in Figure 1.

From Eq. (64), we see that

$$|x| = \sqrt{\frac{13 \pm \sqrt{94}}{5}\frac{p_0}{\rho_0}}$$

are characteristic lines; hence the term

$$\exp\left[-\frac{s\,|x|}{\sqrt{\frac{(13 \mp \sqrt{94})\,p_0}{5\ \ \rho_0}}}\right]$$

shifts whatever occurs at $x = 0$ and $t = 0$ to

$$|x| = \sqrt{\frac{(13 \pm \sqrt{94})\,p_0}{5\ \ \rho_0}}\ t$$

at t; i.e., signals travel along wave fronts.

The second term contributes to the integral

$$\exp\left[-\frac{5/9}{\sqrt{\frac{(13\mp\sqrt{94})}{5}\frac{p_0}{\rho_0}}}\left[\frac{1}{2}-\frac{1}{\sqrt{94}}\right]|x|\right]$$

which is a damping term such that all perturbations induced die out exponentially. The transforms after having being expanded into power series of $1/\lambda$ may be represented in the following form:

$$\bar{Q}=\left[a_0+\frac{a_1}{\lambda}+\frac{a_2}{\lambda^2}+\cdots\right]e^{-\frac{\lambda|x|}{\sqrt{\frac{13\mp\sqrt{94}}{5}\frac{p_0}{\rho_0}}}}\cdot e^{-\frac{5/9\,|x|}{\sqrt{\frac{(13\mp\sqrt{94})}{5}\frac{p_0}{\rho_0}}}\left(\frac{1}{2}-\frac{1}{\sqrt{94}}\right)} \tag{65}$$

A term by term inversion of this transform gives a delta function as the leading term which is the signal initially introduced. The second term gives a unit step function, and from there on, a power series solution valid across the lines of characteristics. Therefore, we have essentially a wave front approximation.

2. Solutions Suitable for Large Values of Time. To evaluate a contour integral, the singularities and branch points of the integrand must be first located in order to understand fully the behavior of the integral. By equating $\lambda_{1,2} = 0$, one finds that the points $\lambda = -1$, $-2/3$, $-5/9$, and 0 are branch points. The point $\lambda = -5/9$ is also an essential singularity. As one can see, they are all located to the left of the imaginary axis in the complex λ-plane. In other words,

they all have negative real parts. If this were not the case, it would mean all quantities diverge with respect to time, and such behavior is physically impossible.

For large values of time, the integral gets the dominating contribution around the algebraically largest branch point, which in our case is the origin; hence, we can expand all transforms in powers of s. As an example, S_F is worked out in detail. From Eq. (48)

$$\bar{S}_F = \frac{(sgn\,x)\left[\left(\frac{33}{25}s+1\right)\lambda_1 - \frac{3}{5}\frac{p_0}{\rho_0}s(s+1)\left(s+\frac{2}{3}\right)\right]}{2\frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)(\lambda_1-\lambda_2)} e^{-\sqrt{\lambda_1}|x|} + \frac{(sgn\,x)\left[\left(\frac{33}{25}s+1\right)\lambda_2 - \frac{3}{5}\frac{p_0}{\rho_0}s(s+1)\left(s+\frac{2}{3}\right)\right]}{2\frac{p_0}{\rho_0}\left(\frac{9}{5}s+1\right)(\lambda_2-\lambda_1)} e^{-\sqrt{\lambda_2}|x|}$$

By keeping only the leading terms of the expansion

$$\bar{S}_F \cong \frac{(sgn\,x)}{5\frac{p_0}{\rho_0}} e^{-\sqrt{\frac{2}{3}\frac{\rho_0}{p_0}}\sqrt{s}|x|} + \frac{(sgn\,x)}{\frac{10}{3}\frac{p_0}{\rho_0}} e^{-\frac{s}{\sqrt{\frac{5}{3}\frac{p_0}{\rho_0}}}\left(1-\frac{7}{10}s\right)|x|} \tag{66}$$

The inverse transform S_F is given by

$$S_F = \frac{(sgn\,x)}{5\frac{p_0}{\rho_0}}\delta_1 + \frac{(sgn\,x)}{\frac{10}{3}\frac{p_0}{\rho_0}}\delta_2 \tag{67}$$

where

$$\delta_1 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{st - \sqrt{\frac{2}{3}\frac{\rho_0}{p_0}}\sqrt{s}|x|}\, ds \tag{68}$$

$$\delta_2 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{st - \frac{s}{\sqrt{\frac{5}{3}\frac{p_0}{\rho_0}}}(1-\frac{7}{10}s)|x|}\, ds \tag{69}$$

The other Q_F physical quantities have leading terms in δ_1 and δ_2, except T_F, which is of higher order. This behavior is shown in Eq. (27) in which $\bar{T} \sim s\bar{p}$ for s small. For the part induced by H, we have given some of the results below. These are:

$$S_H = -\frac{3}{5}\sqrt{\frac{2}{3}\frac{\rho_0}{p_0}}\,\delta_3 + \frac{3}{5}\sqrt{\frac{3}{5}\frac{\rho_0}{p_0}}\,\delta_2 \tag{70}$$

$$\theta_H = \frac{3}{5}\sqrt{\frac{2}{3}\frac{\rho_0}{p_0}}\,\delta_3 - \frac{3}{5}\sqrt{\frac{3}{5}\frac{\rho_0}{p_0}}\,\delta_2 \tag{71}$$

and

$$u_H = (\operatorname{sgn} x)\,\frac{3}{5}\,\delta_2 \tag{72}$$

H and F have different influences on the responses, and especially on the discontinuities (see Section IV). The δ's are evaluated in Appendix II. We have given below the results.

$$\delta_1 = \frac{|x|}{2\sqrt{\pi}\sqrt{\frac{3}{2}\frac{p_0}{\rho_0}}\; t^{3/2}}\; e^{-\frac{x^2}{4t\left(\frac{3}{2}\frac{p_0}{\rho_0}\right)}} \tag{73}$$

$$\delta_2 = \frac{1}{2}\sqrt{10/7}\,\frac{1}{\sqrt{2\pi t}}\left\{ e^{-\frac{10}{7c^2}\frac{(x-ct)^2}{4t}} + e^{-\frac{10}{7c^2}\frac{(x+ct)^2}{4t}} \right\} \tag{74}$$

$$\delta_3 = \frac{1}{\sqrt{\pi t}}\; e^{-\frac{x^2}{4t\left(\frac{3}{2}\frac{p_0}{\rho_0}\right)}} \tag{75}$$

IV. DISCUSSION AND CONCLUSIONS

When $t/t_f << 1$ the solutions of the linearized Grad equations show that the original delta function impulse is propagated along two distinct characteristics, representing a "fast" wave and a "slow" wave, compared with the isentropic sound speed. The amplitude of the impulse (or the energy and momentum contained within it) decays exponentially with distance from the plane of origin of the disturbance, and step-function disturbances with "jumps" across both wave fronts are left behind. This behavior is quite different from that predicted by the Navier-Stokes equations. The responses to the force and heat input functions do not differ in any significant way when $t/t_f << 1$.

When $t/t_f >> 1$, the two distinct wave fronts have disappeared, and a disturbance propagating at the isentropic sound speed is observed, accompanied by viscous-conductive diffusion away from this "front". The width of this dissipative zone grows like $\sqrt{\nu\ t}$, while the perturbation amplitudes decay like $1/\sqrt{\nu\ t}$. Both force and heat input delta functions produce such waves. The amplitude of the pressure, density, temperature, and flow velocity perturbations are all of the same order, while the viscous stress and heat flux are of higher order, as expected.

In addition to these wave fronts, a "wake" is left behind when $t/t_f >> 1$, but the character of this wake for the heat input and force impulse delta functions is quite different. In the case of the heat function, the density and temperature perturbations in the wake are of the classical form

$$\frac{1}{\sqrt{\nu t}}\, e^{-\frac{x^2}{4\nu t}}$$

with maximum amplitude at the plane of origin, containing a constant total "area" or heat quantity at all times. The velocity, pressure, etc., are all of higher order. In the case of the force impulse function, however, the density and temperature perturbations act like $\partial/\partial x$ of the classical delta heat function solutions, which means that the maximum amplitude occurs at a distance from the origin $\sim\sqrt{t}$, and the magnitude of this amplitude $\sim t^{-1}$. The area or heat quantity associated with this disturbance decays like $1/\sqrt{t}$. Apparently the application of the delta force function introduces a kind of heat "dipole", or equal and opposite heating and cooling delta function at the origin. Thus, in the case of the heat input function, the wave front and wake perturbations are equally important; but in the case of the force impulse function, the important part of the disturbance is contained in the waves.

For $t/t_f \ll 1$, Grad's equations furnish a kind of average behavior, as observed in Reference 11 for Rayleigh's problem. It would be desirable to examine the present problem with the aid of a somewhat more sophisticated particle velocity distribution function, in order to account for the fact that particles with velocities faster than the Grad characteristics speeds will carry the disturbance ahead of these "wave fronts". It would also be instructive to study other non-steady flow problems, such as the disturbance produced by the sudden heating of an infinite, stationary flat plate, or the piston problem, which have certain close resemblances to the present problem.

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APPENDIX I

SOME FORMULAS IN CONNECTION WITH THE GREEN'S FUNCTION $G^{(2)}(x, \xi)$

In calculating the fundamental solutions, integrals of the type

$$I_n = \int_{-\infty}^{+\infty} G^{(2)}(x;\xi)\, \frac{d^n\delta(\xi)}{d\xi^n}\, d\xi \qquad n = 0, 1, 2, 3$$

and differentiation of the integrals

$$\frac{dI_n}{dx} = \frac{d}{dx}\int_{-\infty}^{+\infty} G^{(2)}(x;\xi)\, \frac{d^n\delta(\xi)}{d\xi^n}\, d\xi \qquad n = 0, 1, 2, 3$$

appear repeatedly.

In this Appendix, we give the results of these integrals and some useful formulas in connection with the evaluation of these integrals.

$$G^{(2)}(x;\xi) = G^{(2)}(x-\xi) = \frac{1}{2(\lambda_1-\lambda_2)}\left[\frac{1}{\sqrt{\lambda_1}} e^{-\sqrt{\lambda_1}|x-\xi|} - \frac{1}{\sqrt{\lambda_2}} e^{-\sqrt{\lambda_2}|x-\xi|}\right] \tag{A. I-1}$$

$$I_0 = \frac{1}{2(\lambda_1-\lambda_2)}\left[\frac{1}{\sqrt{\lambda_1}} e^{-\sqrt{\lambda_1}|x|} - \frac{1}{\sqrt{\lambda_2}} e^{-\sqrt{\lambda_2}|x|}\right] \tag{A. I-2}$$

$$I_1 = \frac{-(Agn\,x)}{2(\lambda_1-\lambda_2)}\left[e^{-\sqrt{\lambda_1}|x|} - e^{-\sqrt{\lambda_2}|x|}\right] \tag{A. I-3}$$

$$I_2 = \frac{1}{2(\lambda_1 - \lambda_2)}\left[\sqrt{\lambda_1}\, e^{-\sqrt{\lambda_1}|x|} - \sqrt{\lambda_2}\, e^{-\sqrt{\lambda_2}|x|}\right] \tag{A. I-4}$$

$$I_3 = -\frac{(\mathrm{sgn}\, x)}{2(\lambda_1 - \lambda_2)}\left[\lambda_1 e^{-\sqrt{\lambda_1}|x|} - \lambda_2 e^{-\sqrt{\lambda_2}|x|}\right] \tag{A. I-5}$$

$$\frac{dI_{n-1}}{dx} = I_n \qquad n = 1, 2, 3 \tag{A. I-6}$$

$$\frac{dI_3}{dx} = \frac{1}{2(\lambda_1 - \lambda_2)}\left[\lambda_1\sqrt{\lambda_1}\, e^{-\sqrt{\lambda_1}|x|} - \lambda_2\sqrt{\lambda_2}\, e^{-\sqrt{\lambda_2}|x|}\right] \tag{A. I-7}$$

$$\frac{dI_n}{dx} = \frac{d^{n+1} I_0}{dx^{n+1}} \qquad n = 1, 2, 3 \tag{A. I-8}$$

$$(\mathrm{sgn}\, x) = \begin{matrix} +1 & \text{for} & x > 0 \\ -1 & & x < 0 \end{matrix} \tag{A. I-9}$$

$$\frac{d}{dx}(\mathrm{sgn}\, x) = 2\,\delta(x) \tag{A. I-10}$$

$$\frac{d}{dx}|x| = (\mathrm{sgn}\, x) \tag{A. I-11}$$

APPENDIX II

COLLECTED RESULTS OF CONTOUR INTEGRALS

Integral 1

$$\delta_1 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{st - a\sqrt{s}}\, ds \qquad \text{where} \qquad a = \frac{|x|}{\sqrt{\frac{3}{2}\frac{p_0}{\rho_0}}}$$

In this integral, $s = 0$ is a branch point; hence, we may consider the contour as the one given in Figure 2.

δ_1 is equivalent to the integral along path I, but we know that I = - III - IV since II and V vanish as R goes to infinity. If we let $s = re^{i\theta}$, thus

$$s = re^{i\pi} = -r \qquad \text{along III ,} \qquad \text{and}$$

$$s = re^{-i\pi} = -r \qquad \text{along IV}$$

$$\delta_1 = \frac{1}{\pi}\int_0^\infty e^{-rt} \sin a\sqrt{r}\, dr$$

If we introduce the transformation

$$r = \beta^2 \qquad \text{and} \qquad dr = 2\beta d\beta \text{ .}$$

then

$$\delta_1 = \frac{2}{\pi}\int_0^\infty \beta e^{-\beta^2 t} \sin a\beta\, d\beta = -\frac{2}{\pi}\frac{\partial}{\partial a}\int_0^\infty e^{-\beta^2 t}\cos a\beta\, d\beta$$

$$= \frac{|x|}{2\sqrt{\pi}\sqrt{\frac{3}{2}\frac{p_0}{\rho_0}}\, t^{3/2}}\, e^{-\frac{x^2}{4t\sqrt{\frac{3}{2}\frac{p_0}{\rho_0}}}}$$

Integral 2

$$\delta_2 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{st - \frac{s}{\sqrt{\frac{5}{3}\frac{p_0}{\rho_0}}}(1-\frac{7}{10}s)|x|}\, ds$$

We introduce the new variable, $z = s - \frac{5}{7}$

$$\delta_2 = \frac{e^{\frac{5}{7}(t-\frac{|x|}{c})}}{2\pi i}\int_{-i\infty}^{+i\infty} e^{zt + \frac{7}{10}\frac{|x|}{c}z^2}\, dz$$

Along the path $z = iy$; hence

$$\delta_2 = \frac{1}{\pi}\, e^{\frac{5}{7}(t-\frac{|x|}{c})}\int_{-\infty}^{+\infty} e^{iyt - \frac{7}{10}\frac{|x|}{c}y^2}\, dy$$

$$= \frac{1}{2}\sqrt{\frac{10}{7}}\,\frac{1}{\sqrt{2\pi t}}\left\{ e^{-\frac{10}{7c^2}\frac{(x-ct)^2}{4t}} + e^{-\frac{10}{7c^2}\frac{(x+ct)^2}{4t}} \right\}$$

Integral 3

$$\delta_3 = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{st - a\sqrt{s}}\,\frac{ds}{\sqrt{s}} = \frac{1}{\sqrt{\pi t}}\, e^{-\frac{x^2}{\frac{3}{2}\frac{p_0}{\rho_0}4t}}$$

δ_3 is integrated in the same way as δ_1 .

APPENDIX III

SOME THEOREMS ABOUT FUNDAMENTAL SOLUTIONS

In this appendix we shall state some simple theorems with proofs about fundamental solutions of some special linear differential equations.

Theorem 1

If $G_i^{(1)}$ is the fundamental solution of

$$(M - \lambda_i)u = -f(x) \qquad i = 1,\ 2 \tag{A. III-1}$$

defined by

$$u(x) = \int^{\infty} G_i^{(1)}(x;\xi) f(\xi)\, d\xi \tag{A. III-2}$$

where M stands for any linear differential operator in one, two, or three dimensional space, λ_i are constants with the condition $\lambda_1 \neq \lambda_2$ and $\int^{\infty}$ means to integrate over all components of the position vector $\vec{\xi}$.
Then the fundamental solution of

$$(M - \lambda_1)(M - \lambda_2)u = -f(x) \tag{A. III-3}$$

over the same region of the space is given by

$$G^{(2)}(x;\xi) = \frac{1}{\lambda_1 - \lambda_2}\left(G_1^{(1)} - G_2^{(1)}\right) \tag{A. III-4}$$

such that

$$u(x) = \int^{\infty} G^{(2)}(x;\xi) f(\xi)\, d\xi \quad . \tag{A. III-5}$$

Proof

Since M is a linear operator and λ_1, λ_2 are constants, ($\lambda_1 \neq \lambda_2$), the operators $(M - \lambda_1)$ and $(M - \lambda_2)$ are commutable. Thus (A. III-3) may also be written as

$$(M-\lambda_2)(M-\lambda_1)u = -f(x) \quad . \tag{A. III-6}$$

Considering $(M - \lambda_2)u$ as an unknown function in (A. III-3) and a pplying the definition of fundamental solutions given by (A. III-2), we have

$$(M-\lambda_2)u = \iint^{\infty} G_1^{(1)}(x;\xi)f(\xi)d\xi \quad . \tag{A. III-7}$$

Similarly, from Eq. (A. III-6), we obtain

$$(M-\lambda_1)u = \iint^{\infty} G_2^{(1)}(x;\xi)f(\xi)d\xi \quad . \tag{A. III-8}$$

Subtracting Eq. (A. III-8) from Eq. (A. III-7) yields

$$(\lambda_1-\lambda_2) = \iint^{\infty} (G_1^{(1)} - G_2^{(1)})f(\xi)d\xi \quad . \tag{A. III-9}$$

Comparing Eq. (A. III-9) with Eq. (A. III-5) we obtain, for $\lambda_1 \neq \lambda_2$

$$G^{(2)}(x;\xi) = \frac{1}{\lambda_1-\lambda_2}(G_1^{(1)} - G_2^{(1)}) \quad .$$

This proves the theorem.

Theorem 2

Use the same notations and definitions as in theorem 1, for $i = 1, 2, 3$, $\lambda_1 \neq \lambda_2 \neq \lambda_3$. The fundamental solution of

$$(M-\lambda_1)(M-\lambda_2)(M-\lambda_3)u = -f(x) \tag{A. III-10}$$

is given by

$$G^{(3)}(x;\xi) = \frac{G_1^{(1)}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{G_2^{(1)}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{G_3^{(1)}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} \tag{A. III-11}$$

such that

$$u(x) = \int\!\!\int^{\infty} G^{(3)}(x;\xi) f(\xi)\, d\xi$$

<u>Proof</u>

Here operators $(M - \lambda_1)$, $(M - \lambda_2)$ and $(M - \lambda_3)$ are again commutable, so by applying the definitions of $G_i^{(1)}$ given by Eq. (A. III-2) to Eq. (. III-10) we have

$$(M-\lambda_2)(M-\lambda_3)u = [M^2 - (\lambda_2+\lambda_3)M + \lambda_2\lambda_3]u = \int\!\!\int^{\infty} G_1^{(1)} f(\xi)\, d\xi \tag{A. III-12}$$

$$(M-\lambda_3)(M-\lambda_1)u = [M^2 - (\lambda_3+\lambda_1)M + \lambda_3\lambda_1]u = \int\!\!\int^{\infty} G_2^{(1)} f(\xi)\, d\xi \tag{A. III-13}$$

$$(M-\lambda_1)(M-\lambda_2)u = [M^2 - (\lambda_1+\lambda_2)M + \lambda_1\lambda_2]u = \int\!\!\int^{\infty} G_3^{(1)} f(\xi)\, d\xi \quad . \tag{A. III-14}$$

For the case $\lambda_1 \neq \lambda_2 \neq \lambda_3$, the following identities are easily proven:

$$\frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} = 0 \tag{A. III-15}$$

$$\frac{\lambda_2+\lambda_3}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\lambda_3+\lambda_1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{\lambda_1+\lambda_2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} = 0 \tag{A. III-16}$$

and

$$\frac{\lambda_2 \lambda_3}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{\lambda_3 \lambda_1}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} + \frac{\lambda_1 \lambda_2}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} = 1 \quad . \quad \text{(A. III-17)}$$

Divide Eq. (A. III-12) by $(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)$, Eq. (A. III-13) by $(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)$ and Eq. (A. III-14) by $(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)$ and adding, using the relations Eqs. (A. III-15) through (A. III-17), we then obtain

$$u(x) = \iint^{\infty} \left[\frac{G_1^{(1)}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} + \frac{G_2^{(1)}}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} + \frac{G_3^{(1)}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \right] f(\xi)\, d\xi \quad . \quad \text{(A. III-18)}$$

Therefore the result (A. III-11) follows.

APPENDIX IV

SOLUTIONS OF NAVIER-STOKES EQUATIONS AND FOURIER CONDUCTION LAW

The equivalent problem has been treated by T. Y. Wu[8] based on a system of linearized Navier Stokes equations and the Fourier conduction law. The fundamental solutions generated by a unit impulse and a unit heat addition are obtained for the Prandtl number, $Pr = \frac{c_p \mu_o}{K_o}$, taken to be 3/4. The reason for choosing this particular value is to lower the order of one of the differential equations (equation for p). Since at large values of time the solutions of Grad's equations are expected to approach that of the Navier-Stokes, Wu's work is therefore of special interest to us. However, the Prandtl number associated with Grad's equations is 2/3. In order to make any direct comparison, modifications must be made. Here, Wu's problem is reworked without specifying the Prandtl number. The distorted coordinates are introduced and a parallel way of solving the equations is taken to make a step by step comparison with Grad's equations possible. The approximation made for the evaluation of integral transforms are also duplicated for the simple reason that the forms of the integrands obtained from Grad's equations are algebraically more involved, and the present evaluations of the integrals are limited to very rough first approximations.

The following are Wu's original equations in physical coordinates x' and t'.

<u>Continuity</u>

$$\frac{\partial s}{\partial t'} + \frac{\partial u}{\partial x'} = 0 \qquad \text{(A. IV-1)}$$

Momentum

$$\frac{\partial u}{\partial t'} + \frac{p_0}{\rho_0}\frac{\partial p}{\partial x'} - \frac{4\mu}{3}\frac{\partial^2 u}{\partial x'^2} = F' \tag{A. IV-2}$$

Energy

$$\frac{\partial \theta}{\partial t'} - K\frac{\partial^2 \theta}{\partial x'^2} = (\gamma - 1)\frac{\partial s}{\partial t'} + H' \tag{A. IV-3}$$

State

$$p = s + \theta \tag{A. IV-4}$$

where F' is the external force, H' is heat addition, $K = K_o/(c_v \rho_o)$, $\mu = \mu_o/\rho_o$, and $K_o = 15/4\, R\mu_o$. Later on γ is taken to be 5/3 for a monatomic gas.

The above equations can be replaced by an equivalent set of first order partial differential equations similar to Grad's scheme in distorted coordinates x and t introduced previously.

The equivalent set of equations are

Continuity

$$\frac{\partial s}{\partial t} + \frac{\partial u}{\partial x} = 0 \tag{A. IV-5}$$

Momentum

$$\frac{\partial u}{\partial t} + \frac{p_0}{\rho_0}\frac{\partial p}{\partial x} + \frac{1}{\rho_0}\frac{\partial \tau}{\partial x} = F(x,t) \tag{A. IV-6}$$

Energy

$$\frac{\partial p}{\partial t} + \gamma\frac{\partial u}{\partial x} + \frac{(\gamma-1)}{p_0}\frac{\partial q}{\partial x} = H(x,t) \tag{A. IV-7}$$

Stress

$$\frac{4}{3} p_0 \frac{\partial u}{\partial x} + \tau = 0 \tag{A. IV-7}$$

Heat

$$\frac{15}{4} \frac{p_0^2}{\rho_0} \frac{\partial \theta}{\partial x} + q = 0 \tag{A. IV-9}$$

State

$$p = s + \theta \tag{A. IV-10}$$

where $$F' = \frac{p_0}{\mu_0} F \qquad H' = \frac{p_0}{\mu_0} H$$

Equations (A. IV-5), (A. IV-6), and (A. IV-7) are exactly the same as that of Grad's for $\gamma = 5/3$ since they represent nothing but the conservation of mass, momentum, and energy, respectively. The differences show up in the stress and heat conduction relations. The equation of state is of course unchanged. Unlike Grad's system, the above equations are parabolic, and no finite characteristic speeds exist.

If we apply the method of Laplace transform with zero initial conditions to Eqs. (A. IV-5) to (A. IV-10) we obtain the following equations for the transformed quantities.

Continuity

$$\frac{d\bar{u}}{dx} = -s\,\bar{s} \tag{A. IV-11}$$

Momentum

$$s\bar{u} + \frac{p_0}{\rho_0}\frac{d\bar{p}}{dx} + \frac{1}{\rho_0}\frac{d\bar{\tau}}{dx} = \bar{F} \tag{A. IV-12}$$

Energy

$$\frac{d\bar{q}}{dx} = \frac{3p_0}{2}\lambda\left(\frac{5}{3}\bar{S} - \bar{p}\right) + \frac{3p_0}{2}\bar{H} \qquad \text{(A. IV-13)}$$

Stress

$$\bar{\tau} = \frac{4}{3}p_0\lambda\bar{S} \qquad \text{(A. IV-14)}$$

Heat

$$\bar{q} = -\frac{15}{4}\frac{p_0^2}{\rho_0}\frac{d\bar{\theta}}{dx} \qquad \text{(A. IV-15)}$$

State

$$\bar{p} = \bar{S} + \bar{\theta} \quad . \qquad \text{(A. IV-16)}$$

Successive elimination of the transformed variables leads us to a final equation

$$L\,\bar{Q} = \mathcal{J}_{\bar{Q}} \qquad \text{(A. IV-17)}$$

where L is the fourth order differential operator,

$$L = \frac{p_0^2}{\rho_0^2}\left(\frac{4}{3}\lambda+1\right)\frac{d^4}{dx^4} - \frac{p_0}{\rho_0}\frac{\lambda}{3}\left(2+\frac{23}{5}\lambda\right)\frac{d^2}{dx^2} + \frac{2}{5}\lambda^3$$

$\bar{Q}$ stands for any one of the six dependent variables, and

$\mathcal{J}_{\bar{Q}}$ denotes the inhomogeneous part of the equation corresponding to each particular $\bar{Q}$.

The operator L can be factored into two second order operators as follows:

$$L = a\frac{d^4}{dx^4} + 2b\frac{d^2}{dx^2} + c = a\left(\frac{d^2}{dx^2} - \lambda_1\right)\left(\frac{d^2}{dx^2} - \lambda_2\right)$$

where

$$a = \frac{p_0^2}{\rho_0^2}\left(\frac{4}{3}\lambda + 1\right)$$

$$b = -\frac{p_0}{\rho_0}\frac{\lambda}{6}\left(2 + \frac{23}{5}\lambda\right)$$

$$c = \frac{2}{5}\lambda^3$$

and

$$\lambda_{1,2} = \frac{1}{a}\left(-b \pm \sqrt{b^2 - ac}\right)$$

or more precisely,

$$\lambda_{1,2} = \frac{\rho_0}{6p_0}\frac{\lambda}{\left(\frac{4}{3}\lambda+1\right)}\left[2 + \frac{23}{5}\lambda \pm \sqrt{4 + 4\lambda + \frac{49}{25}\lambda^2}\right] .$$

Knowing $\mathcal{J}_{\bar{Q}}$, one can write down immediately the solution for each $\bar{Q}$.

$$\bar{Q} = -\frac{1}{a}\int_{-\infty}^{+\infty} G^{(2)}(x;\xi)\mathcal{J}_{\bar{Q}}(\xi)\,d\xi \qquad \text{(A. IV-18)}$$

Furthermore, since the equations are linear, we can split each solution into two parts, one corresponding to F and the other to H.

$$\bar{Q} = \bar{Q}_F + \bar{Q}_H \quad , \qquad \mathcal{J}_{\bar{Q}} = \mathcal{J}_{\bar{Q}_F} + \mathcal{J}_{\bar{Q}_H}$$

$$\bar{Q}_F = -\frac{1}{a}\int_{-\infty}^{+\infty} G^{(2)}(x;\xi)\mathcal{J}_{\bar{Q}_F}\,d\xi$$

$$Q_H = -\frac{1}{a}\int_{-\infty}^{+\infty} G^{(2)}(x;\xi)\mathcal{J}_{\bar{Q}_H}\,d\xi$$

where

$$G^{(2)}(x;\xi) = \frac{1}{2(\lambda_1 - \lambda_2)}\left(\frac{1}{\sqrt{\lambda_1}}e^{-\sqrt{\lambda_1}|x|} - \frac{1}{\sqrt{\lambda_2}}e^{-\sqrt{\lambda_2}|x|}\right)$$

is the Green's function of the operator L. The transforms are given below

$$\bar{S}_F = \frac{(\mathrm{sgn}\,x)\left(\lambda_1 - \frac{2}{5} s\, \rho_0/p_0\right) e^{-\sqrt{\lambda_1}|x|}}{2 \frac{p_0}{\rho_0}\left(\frac{4}{3}s+1\right)(\lambda_1-\lambda_2)} + \frac{(\mathrm{sgn}\,x)\left(\lambda_2 - \frac{2}{5} s\, \rho_0/p_0\right) e^{-\sqrt{\lambda_2}|x|}}{2 \frac{p_0}{\rho_0}\left(\frac{4}{3}s+1\right)(\lambda_2-\lambda_1)} \qquad \text{(A. IV-19)}$$

$$\bar{P}_F = \frac{(\mathrm{sgn}\,x)\left(\lambda_1 - \frac{2}{3} s\, \rho_0/p_0\right) e^{-\sqrt{\lambda_1}|x|}}{2 \frac{p_0}{\rho_0}\left(\frac{4}{3}s+1\right)(\lambda_1-\lambda_2)} + \frac{(\mathrm{sgn}\,x)\left(\lambda_2 - \frac{2}{3} s\, \rho_0/p_0\right) e^{-\sqrt{\lambda_2}|x|}}{2 \frac{p_0}{\rho_0}\left(\frac{4}{3}s+1\right)(\lambda_2-\lambda_1)} \qquad \text{(A. IV-20)}$$

$$\bar{\theta}_F = -\frac{4}{15} \frac{(\mathrm{sgn}\,x)\, s\, \rho_0/p_0\, e^{-\sqrt{\lambda_1}|x|}}{2 \frac{p_0}{\rho_0}\left(\frac{4}{3}s+1\right)(\lambda_1-\lambda_2)} - \frac{4}{15} \frac{(\mathrm{sgn}\,x)\, s\, \rho_0/p_0\, e^{-\sqrt{\lambda_2}|x|}}{2 \frac{p_0}{\rho_0}\left(\frac{4}{3}s+1\right)(\lambda_2-\lambda_1)} \qquad \text{(A. IV-21)}$$

$$\bar{\tau}_F = \frac{(\mathrm{sgn}\,x)\, \frac{4}{3} p_0\, s \left(\lambda_1 - \frac{2}{5} s\, \rho_0/p_0\right) e^{-\sqrt{\lambda_1}|x|}}{2 \frac{p_0}{\rho_0}\left(\frac{4}{3}s+1\right)(\lambda_1-\lambda_2)} + \frac{(\mathrm{sgn}\,x)\, \frac{4}{3} p_0\, s \left(\lambda_2 - \frac{2}{5} s\, \rho_0/p_0\right) e^{-\sqrt{\lambda_2}|x|}}{2 \frac{p_0}{\rho_0}\left(\frac{4}{3}s+1\right)(\lambda_2-\lambda_1)} \qquad \text{(A. IV-22)}$$

$$\bar{q}_F = -\frac{\rho_0\, s \sqrt{\lambda_1}\, e^{-\sqrt{\lambda_1}|x|}}{2\left(\frac{4}{3}s+1\right)(\lambda_1-\lambda_2)} - \frac{\rho_0\, s \sqrt{\lambda_2}\, e^{-\sqrt{\lambda_2}|x|}}{2\left(\frac{4}{3}s+1\right)(\lambda_2-\lambda_1)} \qquad \text{(A. IV-23)}$$

$$\bar{u}_F = \frac{\lambda_1 \sqrt{\lambda_1}\, e^{-\sqrt{\lambda_1}|x|}}{2 s (\lambda_1-\lambda_2)} - \frac{\frac{2}{3}\left(1+\frac{4}{5}s\right)\sqrt{\lambda_1}\, e^{-\sqrt{\lambda_1}|x|}}{2 \frac{p_0}{\rho_0}\left(\frac{4}{3}s+1\right)(\lambda_1-\lambda_2)} \qquad \text{(A. IV-24)}$$

$$+ \frac{\lambda_2 \sqrt{\lambda_2}\, e^{-\sqrt{\lambda_2}|x|}}{2 s (\lambda_2-\lambda_1)} - \frac{\frac{2}{3}\left(1+\frac{4}{5}s\right)\sqrt{\lambda_2}\, e^{-\sqrt{\lambda_2}|x|}}{2 \frac{p_0}{\rho_0}\left(\frac{4}{3}s+1\right)(\lambda_2-\lambda_1)}$$

$$\bar{s}_H = \frac{-\frac{2}{5}\sqrt{\lambda_1}\, e^{-\sqrt{\lambda_1}|x|}}{2\frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_1-\lambda_2)} - \frac{\frac{2}{5}\sqrt{\lambda_2}\, e^{-\sqrt{\lambda_2}|x|}}{2\frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_2-\lambda_1)} \qquad \text{(A. IV-25)}$$

$$\bar{P}_H = \frac{(\frac{8}{15}s\sqrt{\lambda_1} - \frac{2}{5}s^2\frac{\rho_0}{p_0}\frac{1}{\sqrt{\lambda_1}})e^{-\sqrt{\lambda_1}|x|}}{2\frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_1-\lambda_2)} + \frac{(\frac{8}{15}s\sqrt{\lambda_2} - \frac{2}{5}s^2\frac{\rho_0}{p_0}\frac{1}{\sqrt{\lambda_2}})e^{-\sqrt{\lambda_2}|x|}}{2\frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_2-\lambda_1)} \qquad \text{(A. IV-26)}$$

$$\bar{\theta}_H = \frac{\frac{2}{5}\sqrt{\lambda_1}}{2\frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_1-\lambda_2)}\left[1+\frac{4}{3}s-\frac{2}{5}s^2\frac{\rho_0}{p_0}\frac{1}{\lambda_1}\right]e^{-\sqrt{\lambda_1}|x|} + \frac{\frac{2}{5}\sqrt{\lambda_2}}{2\frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_2-\lambda_1)}\left[1+\frac{4}{3}s-\frac{2}{5}s^2\frac{\rho_0}{p_0}\frac{1}{\lambda_2}\right]e^{-\sqrt{\lambda_2}|x|} \qquad \text{(A. IV-27)}$$

$$\bar{\tau}_H = -\frac{8}{15}\frac{p_0 s\sqrt{\lambda_1}\, e^{-\sqrt{\lambda_1}|x|}}{2\frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_1-\lambda_2)} - \frac{8}{15}\frac{p_0 s\sqrt{\lambda_2}\, e^{-\sqrt{\lambda_2}|x|}}{2\frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_2-\lambda_1)} \qquad \text{(A. IV-28)}$$

$$\bar{q}_H = \frac{\frac{3}{2}(\mathrm{sgn}\,x)}{2(\lambda_1-\lambda_2)}\left[p_0\lambda_1 - \frac{\rho_0 s^2}{(\frac{4}{3}s+1)}\right]e^{-\sqrt{\lambda_1}|x|} + \frac{\frac{3}{2}(\mathrm{sgn}\,x)}{2(\lambda_2-\lambda_1)}\left[p_0\lambda_2 - \frac{\rho_0 s^2}{(\frac{4}{3}s+1)}\right]e^{-\sqrt{\lambda_2}|x|} \qquad \text{(A. IV-29)}$$

$$\bar{u}_H = -\frac{\frac{2}{5}(\mathrm{sgn}\,x)\,s}{2\frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_1-\lambda_2)}e^{-\sqrt{\lambda_1}|x|} - \frac{\frac{2}{5}(\mathrm{sgn}\,x)\,s}{2\frac{p_0}{\rho_0}(\frac{4}{3}s+1)(\lambda_2-\lambda_1)}e^{-\sqrt{\lambda_2}|x|} \qquad \text{(A. IV-30)}$$

The Inverse Transforms

The inverse transform of a quantity Q is given by

$$Q = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{st}\,\bar{Q}(s)\,ds \tag{A. IV-31}$$

where σ is to the right of all singularities. Therefore, it is important to understand the behavior of $\bar{Q}$, or more precisely, the behavior of λ s in this case. We have

$$\lambda_{1,2} = \frac{1}{6\,\frac{p_0}{\rho_0}}\,\frac{s}{(\frac{4}{3}s+1)}\left(2 + \frac{23}{5}s \pm 2\sqrt{1+s+\frac{49}{100}s^2}\right) \quad . \tag{A. IV-32}$$

By setting $\lambda_{1,2} = 0$, one finds that $s = 0$ and $s = -3/4$ are branch points, and the latter is also being an essential singularity. Since we are mainly interested here in the value of the integrals at large time, a branch point approximation would serve the purpose. The point $s = 0$ undoubtedly plays the dominating part; hence we will focus our attention at this point. Given below is the case of S_F worked out in detail. The rest of the integrals are treated in similar fashion.

The Evaluation of S_F for Large t

If we substitute the λ s into S_F and expand it in powers of s, we get, by keeping only the leading terms

$$\bar{S}_F \cong \frac{(sgn\,x)\,\frac{2}{5}\,e^{-\sqrt{\lambda_1^*}|x|}}{2\,\frac{p_0}{\rho_0}} + \frac{(sgn\,x)\,3/5\,e^{-\sqrt{\lambda_2^*}|x|}}{2\,\frac{p_0}{\rho_0}} \quad , \tag{A. IV-33}$$

where

$$\lambda_1^* = \frac{2}{3}\,\frac{s}{p_0/\rho_0} \quad , \quad \lambda_2^* = \frac{s^2}{5/3\,p_0/\rho_0}\left(1 - \frac{7}{5}s\right)$$

are obtained by taking $\sqrt{1+s+49s^2/100} = 1 + \frac{1}{2}(s + \frac{49}{100}s^2) + \cdots$

Therefore, we have

$$\bar{S}_F \cong \frac{(\Delta g n x)}{5\frac{p_0}{\rho_0}} e^{-\sqrt{\frac{2\rho_0}{3p_0}}\sqrt{s}|x|} + \frac{(\Delta g n x)3}{10\frac{p_0}{\rho_0}} e^{-\sqrt{\frac{3\rho_0}{5p_0}}s(1-\frac{7}{10}s)|x|} \quad . \qquad \text{(A. IV-33a)}$$

The inverse transform of $\bar{S}_F$ consists of two integrals δ_1 and δ_2, where

$$S_F = \frac{(\Delta g n x)}{5\frac{p_0}{\rho_0}}\delta_1 + \frac{(\Delta g n x)}{\frac{10}{3}\frac{p_0}{\rho_0}}\delta_2 \quad , \qquad \text{(A. IV-34)}$$

$$\delta_1 = \frac{|x|}{2\sqrt{\pi}\sqrt{\frac{3}{2}\frac{p_0}{\rho_0}}\, t^{3/2}} e^{-\frac{x^2}{4t\sqrt{\frac{3}{2}p_0/\rho_0}}}$$

$$\delta_2 = \frac{1}{2}\sqrt{\frac{10}{7}}\frac{1}{\sqrt{2\pi t}}\left\{ e^{-\frac{10}{7c^2}\frac{(x-ct)^2}{4t}} + e^{-\frac{10}{7c^2}\frac{(x+ct)^2}{4t}} \right\} \quad .$$

The details of evaluation of δ_1 and δ_2 are given in Appendix II. We find also that, by keeping the leading terms

$$P_F = \frac{(\Delta g n x)}{2\frac{p_0}{\rho_0}}\delta_2 \qquad \text{(A. IV-35)}$$

$$\theta_F = -\frac{(\Delta g n x)}{5\frac{p_0}{\rho_0}}\delta_1 + \frac{(\Delta g n x)}{5\frac{p_0}{\rho_0}}\delta_2 \qquad \text{(A. IV-36)}$$

$$\bar{\tau}_F \sim s\bar{P}_F \quad . \qquad \text{(A. IV-37)}$$

Therefore τ_F is of higher order compared with P_F at large t (small s).

$$u_F = \frac{1}{4} \frac{\delta_2}{\sqrt{5/3\ p_0/\rho_0}} \tag{A. IV-38}$$

$$q_F = \frac{3}{8} \frac{p_0 \delta_2}{\sqrt{5/3\ p_0/\rho_0}} \tag{A. IV-39}$$

Some of the responses induced by H are given below

$$\bar{s}_H \cong -\frac{3}{10} \frac{1}{\sqrt{3/2\, p_0/\rho_0}} \frac{e^{-\sqrt{\lambda_1^*}|x|}}{\sqrt{s}} + \frac{3}{10c}\, e^{-\sqrt{\lambda_2^*}|x|} \tag{A. IV-40}$$

$$\bar{\theta}_H \cong \frac{3}{10} \frac{1}{\sqrt{3/2\, p_0/\rho_0}} \frac{e^{-\sqrt{\lambda_1^*}|x|}}{\sqrt{s}} - \frac{3}{10c}\, e^{-\sqrt{\lambda_2^*}|x|} \tag{A. IV-41}$$

$$\bar{u}_H = -(\mathrm{sgn}x)\ 3/_{10}\ e^{-\sqrt{\lambda_2^*}|x|} \tag{A. IV-42}$$

Hence

$$s_H = -\frac{3}{10} \frac{1}{\sqrt{\frac{3}{2} p_0/\rho_0}} \delta_3 + \frac{3}{10c} \delta_2 \tag{A. IV-43}$$

$$\theta_H = -s_H \tag{A. IV-44}$$

and

$$u_H = (\mathrm{sgn}x) \frac{3}{10} \delta_2 \tag{A. IV-45}$$

where

$$\delta_3 = \frac{1}{\sqrt{\pi t}}\, e^{-\frac{x^2}{\frac{3}{2}\frac{p_0}{\rho_0} 4t}} \tag{A. IV-46}$$

δ_1 and δ_3 represent the wake, while δ_2 represents the two running sound waves.

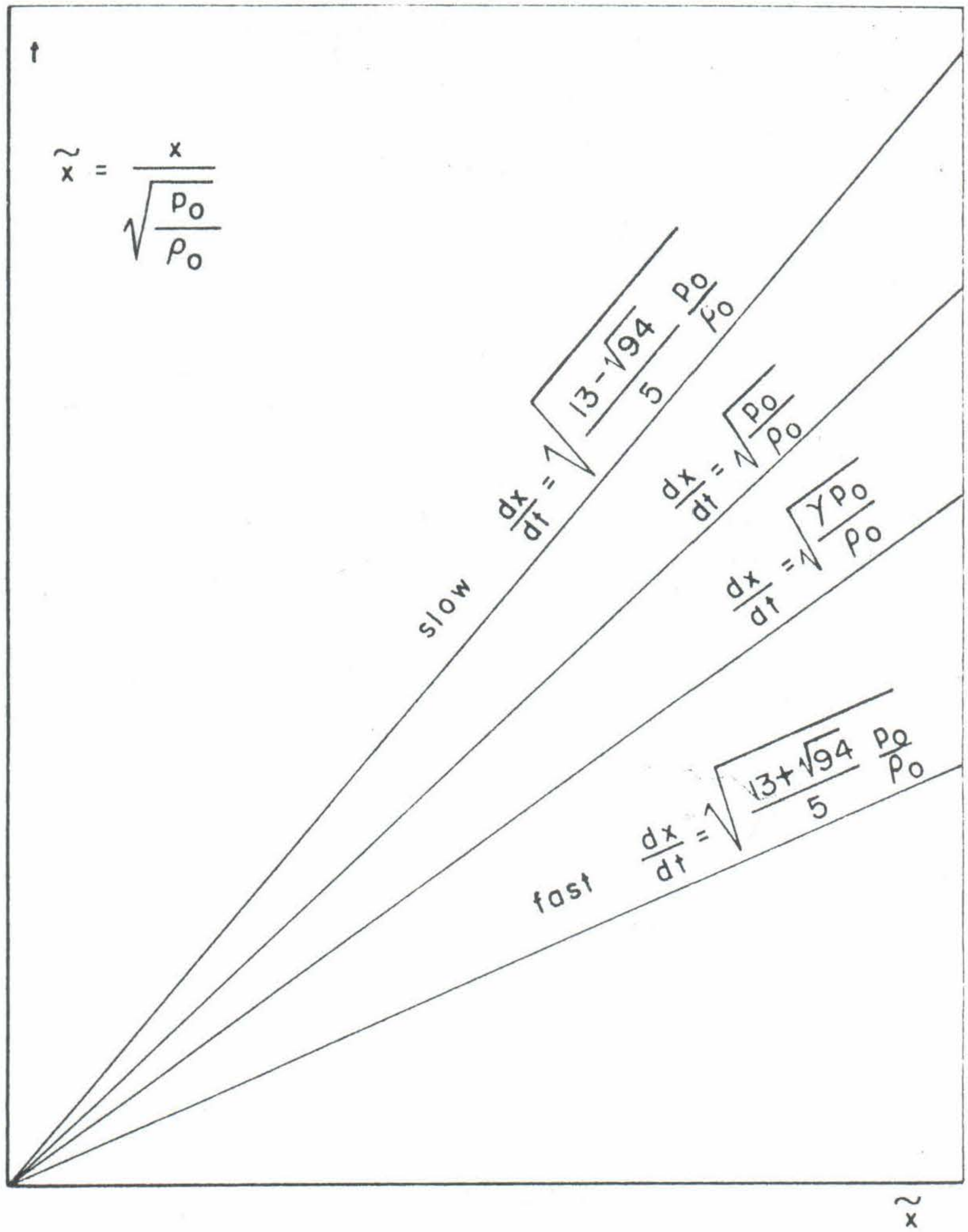

FIG. I — CHARACTERISTICS NORMALIZED AGAINST THE ISOTHERMAL SPEED OF SOUND

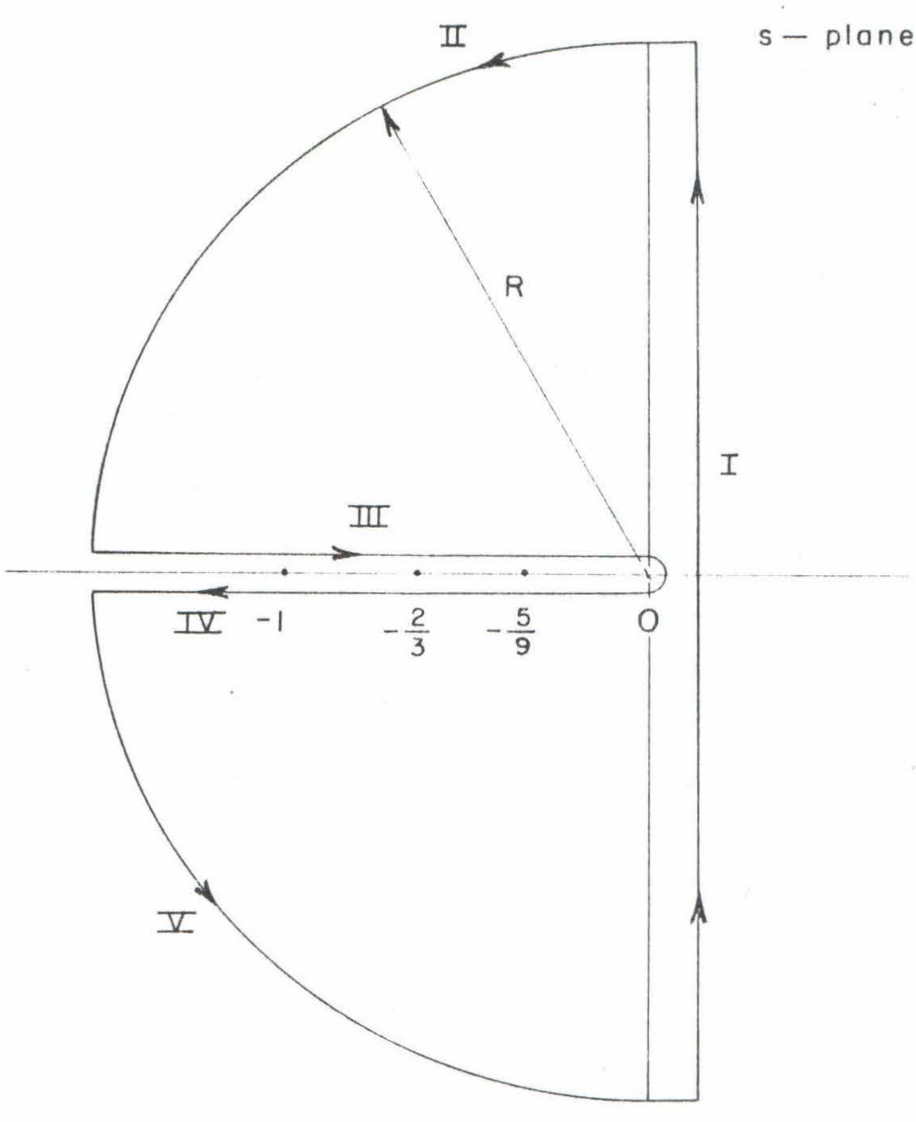

FIG. 2

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